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The structures and interactions of solitary waves in a (2+1)-dimensional coupled nonlinear extension of the reaction-diffusion equation

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Abstract
The complete integrability of a (2+1)-dimensional coupled nonlinear extension of the reaction-diffusion (CNLERD) equation using the technique of Painlev*é*(P)-analysis is investigated. Using the formalism of Weiss *et al* (1983 *J. Math. Phys.* **24** 522), the arbitrariness of the expansion coefficients are proved. Besides, following the Hirota's formalism (Hirota R 1980 Direct methods in soliton theory *Soliton* (Berlin: Springer)) combined to Weiss *et al*'s methodology (Weiss *et al* 1984 *J. Math. Phys.* **25** 13), the consistency of the truncation is shown. Thus, without the use of Kruskal's simplification, the Bäcklund transformation (BT) of the equations is obtained via the truncation procedure. Taking into account the arbitrariness of the expansion coefficients in the foregoing truncation, a typical spectrum of localized coherent structures may be unearthed. The scattering behavior of such structures is also investigated.

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1. Introduction

Modern soliton theory may be widely applied in almost all the physics fields [1–10]. Owing to the fact that the role played by integrable models in the soliton theory is important, to find various integrable models may be one of the important problems. Some useful methods have been recently developed in order to deal with these problems. For instance,

- various finite dimensional Hamiltonian systems may be obtained from the symmetry constraint equations of the (1+1)-dimensional integrable partial differential equations [11–16];
- the constraints of some (2+1)-dimensional integrable equations [17–19] may lead to some (1+1)-dimensional integrable models;
- using an asymptotical exact reduction method based on the Fourier expansion from some known integrable equations [20–23], many integrable systems of the same dimensions as that of the original models may be obtained;
- by means of some techniques such as the conformal invariance of the Schwartz of the known integrable models [24], the deformation theory [25, 26], the general Virasoro symmetry algebra [27–29], the (1+1)-dimensional strong symmetry [30–33], some types of higher dimensional integrable models may be found;
- the 'prolongation structure' [34] which has been used in a (1+1)-dimensional reaction-diffusion system by Beccaria and Soliani may be a powerful tool for obtaining some essential properties of nonlinear integrable systems, but also may be helpful in generating some (2+1)-dimensional integrable systems [35].

Furthermore, some powerful methods of investigation of the exact solutions of some (2+1)-dimensional nonlinear partial differential (NLPD) equations, such as the 'general projective Riccati equation method' (GPREM) [36, 37] and the 'multilinear variable separation approach' (MLVSA) [38], have been developed. The second method, that is the MLVSA, has been extended to its 'universal' formula to a great number of (2+1)-dimensional NLPD equations such as the asymmetric Davey–Stewartson equation [39], the dispersive long wave equation [40], the Broer–Kaup–Kupershmidt system [41], among many others. One of the most powerful methods to prove the integration of a model equation is the so-called P-analysis [42, 43] which has been applied to many systems [44–47]. This method is also useful in finding some exact solutions no matter whether the model is integrable or not [48]. One of the major step of the MLVSA may be the P-analysis which may help generating some arbitrary functions, useful in construction of quite rich localized excitations such as solitoffs, compactons, ring solitons, breathers, instantons and others [49]. This method may also be discussed in some other equations by investigating periodic-wave structures like periodic-compacton interaction waves, periodic-kink interaction waves, etc [50].

Recently, while investigating an integrable (2+1)-dimensional (modified) Heisenberg ferromagnet (HF) model [51] using the prolongation structure theory, Zhai *et al* [35] have constructed its corresponding geometrical equivalent counterparts, such as the (2+1)-dimensional nonlinear Schrödinger equation and the coupled (2+1)-dimensional integrable equations, presented through the motion of Minkowski space curves endowed with an additional spatial variable. These last coupled (2+1)-dimensional integrable equations may be given by [35]

$$\hat{\psi}_t + \hat{\psi}_{xy} - \hat{\gamma}\hat{\psi} = 0, \qquad \hat{\phi}_t - \hat{\phi}_{xy} + \hat{\gamma}\hat{\phi} = 0, \qquad \hat{\gamma}_x + (\hat{\phi}\hat{\psi})_y = 0, \tag{1}$$

where $\hat{\psi}, \hat{\phi}$ and $\hat{\gamma}$ are physical observables and subscripts denote partial differentiation. Another physical application of equation (1) has been pointed out by Duan *et al* [52] while presenting equation (1) as a corresponding geometric equivalent (2+1)-dimensional CNLERD equation of the integrable (2+1)-dimensional (modified) HF model. Even though the prolongation structure has the merit that it is not only the covariant geometry theory, but also a powerful way to obtain many properties such as BT, Lax pair, inverse scattering transform, equation (1) may store another interesting fauna of exotic solutions depicted by means of some interesting techniques such as the P-analysis, GPREM [36, 37] and MLVSA [38], just to name a few. Thus, the analysis of the mathematical properties of equation (1) may deserve great interests. These properties may be summarized by the question of its integrability and the construction of its exotic localized excitations followed by a survey of their scattering behavior. In this paper, we aim to provide detail on the scattering properties of such structures by means of the P-analysis.

The paper is organized as follows. In section 2, we present the P-integrability of equation (1) using the formalism of Weiss, Tabor and Carnevale [42, 43]. In section 3, the BT and Hirota's bilinearization properties of this equation are investigated and it is shown that equation (1) may be completely integrable. Following these results, in section 4, we construct some interesting localized excitations, and we survey their scattering behavior. Finally, in section 5, we end with a brief summary of the work.

2. P-integrability of the (2+1)-dimensional CNLERD equation

As usual, we take the following Laurent expansion of the functions $\hat{\psi}, \hat{\phi}$ and $\hat{\gamma}$ about the singular manifold g as follows:

$$\hat{\psi} = \sum_{k=0}^{\infty} \hat{\psi}_k g^{k+\alpha}, \qquad \hat{\phi} = \sum_{k=0}^{\infty} \hat{\phi}_k g^{k+\beta}, \qquad \hat{\gamma} = \sum_{k=0}^{\infty} \hat{\gamma}_k g^{k+\vartheta}. \tag{2}$$

For the leading order analysis, we truncate the previous series given by equation (2) to the zeroth order, and then replace them into equation (1). We may find only one possible branch

$$\alpha = \beta = -1, \qquad \vartheta = -2, \tag{3}$$

and

$$\hat{\psi}_0 \hat{\phi}_0 = -2g_x^2, \qquad \hat{\gamma}_0 = 2g_x g_y. \tag{4}$$

From equation (4), one of the three functions $\hat{\psi}_0$, $\hat{\phi}_0$ and $\hat{\gamma}_0$ may be arbitrary.

In order to obtain the recursion relations, we may substitute equations (2), (3) and (4) into (1). We get

$$\hat{\mathcal{M}}_k \hat{\mathcal{V}}_k = \hat{\mathcal{T}}_k, \tag{5}$$

where $\hat{\mathcal{M}}_k$ is a square matrix, $\hat{\mathcal{V}}_k = (\hat{\psi}_k, \hat{\phi}_k, \hat{\gamma}_k)^T$ and $\hat{\mathcal{T}}_k = (\hat{P}_k, \hat{Q}_k, \hat{U}_k)^T$ with

$$\begin{aligned} \hat{P}_k = {} & \sum_{j=1}^{k-1} \hat{\gamma}_{k-j} \hat{\psi}_j - \hat{\psi}_{k-2,xy} - \hat{\psi}_{k-2,t} \\ & - (k-2)(\hat{\psi}_{k-1} g_t + \hat{\psi}_{k-1,x} g_y + \hat{\psi}_{k-1,y} g_x + \hat{\psi}_{k-1} g_{xy}), \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{Q}_k = {} & \sum_{j=1}^{k-1} \hat{\gamma}_{k-j} \hat{\phi}_j - \hat{\phi}_{k-2,xy} + \hat{\phi}_{k-2,t} \\ & - (k-2)(-\hat{\phi}_{k-1} g_t + \hat{\phi}_{k-1,x} g_y + \hat{\phi}_{k-1,y} g_x + \hat{\phi}_{k-1} g_{xy}), \end{aligned} \tag{7}$$

and

$$\hat{U}_k = - \left[\sum_{j=0}^{k-1} (\hat{\psi}_{k-j-1} \hat{\phi}_j)_y + (k-2) \sum_{j=1}^{k-1} \hat{\psi}_j \hat{\phi}_{k-j} g_y + \hat{\gamma}_{k-1,x} \right]. \tag{8}$$

The matrix $\hat{\mathcal{M}}_k$ is given by

$$\hat{\mathcal{M}}_k = \begin{bmatrix} A_{1k} & A_{2k} & A_{3k} \\ B_{1k} & B_{2k} & B_{3k} \\ C_{1k} & C_{2k} & C_{3k} \end{bmatrix}, \tag{9}$$

with

$$\begin{array}{lll} A_{1k}=k(k-3)g_xg_y, & A_{2k}=0, & A_{3k}=-\hat{\psi}_0, \\ B_{1k}=0, & B_{2k}=A_{1k}, & B_{3k}=-\hat{\phi}_0, \\ C_{1k}=(k-2)\hat{\phi}_0g_y, & C_{2k}=(k-2)\hat{\psi}_0g_y, & C_{3k}=(k-2)g_x. \end{array} \tag{10}$$

Thus, the determinant $\hat{\Delta}_k$ of the matrix $\hat{\mathcal{M}}_k$ is given by

$$\hat{\Delta}_k=k(k-2)(k-3)(k-4)(k+1)g_y^2g_x^3. \tag{11}$$

The resonances may then be found at

$$k=-1,0,2,3,4. \tag{12}$$

The resonance at $k=-1$ corresponds to that of the singularity manifold g being arbitrary.

From the leading order analysis, we know that the resonance at $k=0$ is satisfied identically and one of $\hat{\psi}_0$, $\hat{\phi}_0$ and $\hat{\gamma}_0$, is arbitrary.

For $k=1$, $\hat{\psi}_1$, $\hat{\phi}_1$ and $\hat{\gamma}_1$ are uniquely expressed as follows:

$$\begin{aligned} \hat{\psi}_1&=-\frac{2g_xg_t\hat{\psi}_0+2g_xg_y\hat{\psi}_{0,x}+\hat{\psi}_0^2\hat{\phi}_{0,y}/2+\hat{\psi}_{0,y}g_x^2}{4g_yg_x^2}, \\ \hat{\phi}_1&=-\frac{2g_xg_y\hat{\phi}_{0,x}+\hat{\phi}_0^2\hat{\psi}_{0,y}/2+\hat{\phi}_{0,y}g_x^2-2g_xg_t\hat{\phi}_0}{4g_yg_x^2}, \\ \hat{\gamma}_1&=-2g_{xy}. \end{aligned} \tag{13}$$

For $k=2$, we get the following system,

$$\begin{aligned} 2g_xg_y\hat{\psi}_2+\hat{\psi}_0\hat{\gamma}_2&=\hat{\psi}_{0,t}+\hat{\psi}_{0,xy}-\hat{\psi}_1\hat{\gamma}_1, \\ 2g_xg_y\hat{\phi}_2+\hat{\phi}_0\hat{\gamma}_2&=\hat{\phi}_{0,xy}-\hat{\phi}_{0,t}-\hat{\phi}_1\hat{\gamma}_1, \end{aligned} \tag{14}$$

provided

$$\hat{\gamma}_{1,x}+(\hat{\psi}_1\hat{\phi}_0+\hat{\phi}_1\hat{\psi}_0)_y=0, \tag{15}$$

holds, as it is actually the case. Therefore, one of $\hat{\psi}_2$, $\hat{\phi}_2$ and $\hat{\gamma}_2$, is arbitrary.

For $k=3$, the system (5)–(10) gives

$$\hat{\psi}_0\hat{\gamma}_3=-\hat{P}_3, \qquad \hat{\phi}_0\hat{\gamma}_3=-\hat{Q}_3, \qquad g_y(\hat{\phi}_0\hat{\psi}_3+\hat{\psi}_0\hat{\phi}_3)+g_x\hat{\gamma}_3=\hat{U}_3, \tag{16}$$

where

$$\hat{P}_3=\hat{\gamma}_2\hat{\psi}_1+\hat{\gamma}_1\hat{\psi}_2-\hat{\psi}_{1,xy}-\hat{\psi}_{1,t}-(\hat{\psi}_2g_t+\hat{\psi}_{2,x}g_y+\hat{\psi}_{2,y}g_x+\hat{\psi}_2g_{xy}), \tag{17}$$

$$\hat{Q}_3=\hat{\gamma}_2\hat{\phi}_1+\hat{\gamma}_1\hat{\phi}_2-\hat{\phi}_{1,xy}+\hat{\phi}_{1,t}-(-\hat{\phi}_2g_t+\hat{\phi}_{2,x}g_y+\hat{\phi}_{2,y}g_x+\hat{\phi}_2g_{xy}), \tag{18}$$

and

$$\hat{U}_3=-[\hat{\gamma}_{2,x}+(\hat{\psi}_2\hat{\phi}_0+\hat{\psi}_1\hat{\phi}_1+\hat{\psi}_0\hat{\phi}_2)_y+(\hat{\psi}_1\hat{\phi}_2+\hat{\psi}_2\hat{\phi}_1)g_y]. \tag{19}$$

From equation (14), we may express $\hat{\psi}_2$ and $\hat{\phi}_2$ as a function of $\hat{\gamma}_2$. Then using equations (13) and (4), after some tedious calculations, we may find that

$$\hat{\phi}_0\hat{P}_3-\hat{\psi}_0\hat{Q}_3=0. \tag{20}$$

Thus, from equations (16) and (20), one of the variables $\hat{\psi}_3$, $\hat{\phi}_3$ and $\hat{\gamma}_3$ may be arbitrary.

Finally, for $k=4$, we may derive

$$\begin{aligned} 4g_xg_y\hat{\psi}_4-\hat{\psi}_0\hat{\gamma}_4&=\hat{P}_4, \\ 4g_xg_y\hat{\phi}_4-\hat{\psi}_0\hat{\gamma}_4&=\hat{Q}_4, \\ g_y\hat{\phi}_0\hat{\psi}_4+g_y\hat{\psi}_0\hat{\phi}_4+g_x\hat{\gamma}_4&=\hat{U}_4/2, \end{aligned} \tag{21}$$

with

$$\hat{P}_4 = \hat{\gamma}_3\hat{\psi}_1 + \hat{\gamma}_2\hat{\psi}_2 + \hat{\gamma}_1\hat{\psi}_3 - \hat{\psi}_{2,xy} - \hat{\psi}_{2,t} - 2(\hat{\psi}_3 g_t + \hat{\psi}_{3,x} g_y + \hat{\psi}_{3,y} g_x + \hat{\psi}_3 g_{xy}), \tag{22}$$

$$\hat{Q}_4 = \hat{\gamma}_3\hat{\phi}_1 + \hat{\gamma}_2\hat{\phi}_2 + \hat{\gamma}_1\hat{\phi}_3 - \hat{\phi}_{2,xy} + \hat{\phi}_{2,t} - 2(\hat{\phi}_3 g_t + \hat{\phi}_{3,x} g_y + \hat{\phi}_{3,y} g_x + \hat{\phi}_3 g_{xy}), \tag{23}$$

and

$$\hat{U}_4 = -[\hat{\gamma}_{3,x} + (\hat{\psi}_3\hat{\phi}_0 + \hat{\psi}_2\hat{\phi}_1 + \hat{\psi}_1\hat{\phi}_2 + \hat{\psi}_0\hat{\phi}_3)_y + 2(\hat{\psi}_3\hat{\phi}_1 + \hat{\psi}_2\hat{\phi}_2 + \hat{\psi}_1\hat{\phi}_3)g_y]. \tag{24}$$

From equation (16), it is possible to express for example $\hat{\psi}_3$ versus $\hat{\phi}_3$ and $\hat{\gamma}_3 = -\hat{P}_3/\hat{\psi}_0$. Besides, from equation (21), we may express $\hat{\psi}_4$ and $\hat{\phi}_4$ versus $\hat{\gamma}_4$. With these results, we follow the same procedure as previously done for the case $k = 3$ by replacing $\hat{\psi}_4, \hat{\phi}_4, \hat{\psi}_3, \hat{\gamma}_3, \hat{\psi}_2, \hat{\phi}_2, \hat{\psi}_1, \hat{\phi}_1$ and $\hat{\gamma}_1$ into the last equation of the system (21), which after some tedious calculations, may be satisfied identically. Thus, it comes from equation (21) that one of $\hat{\psi}_4, \hat{\phi}_4$ and $\hat{\gamma}_4$ is arbitrary.

The (2+1)-dimensional CNLERD equation may possess a sufficient number of arbitrary functions. Therefore, this system is P-integrable. It is well known that the P-analysis may also be used to obtain other interesting properties [42, 43]. The complete integrability of the (2+1)-dimensional CNLERD equation may be established if some essential properties such as BT and Hirota's bilinearization [53–55] are proved to exist. In the following section, we may use the truncated P-expansion to derive the BT and Hirota's bilinearization [53–55] of the (2+1)-dimensional CNLERD equation.

3. BT and Hirota's bilinearization properties of the (2+1)-dimensional CNLERD equation

If we take,

$$\hat{\psi}_k = \hat{\phi}_k = \hat{\gamma}_{k+1} = 0, \qquad k \geqslant 2, \tag{25}$$

equation (2) becomes the following truncated expansion

$$\hat{\psi} = \hat{\psi}_0/g + \hat{\psi}_1, \qquad \hat{\phi} = \hat{\phi}_0/g + \hat{\phi}_1, \qquad \hat{\gamma} = \hat{\gamma}_0/g^2 + \hat{\gamma}_1/g + \hat{\gamma}_2. \tag{26}$$

Substituting equation (26) into (1), or rather replacing equation (25) into equations (14)–(24), we straightforwardly derive

$$\hat{\psi}_0\hat{\gamma}_2 = \hat{\psi}_{0,t} + \hat{\psi}_{0,xy} - \hat{\psi}_1\hat{\gamma}_1, \qquad \hat{\phi}_0\hat{\gamma}_2 = -\hat{\phi}_{0,t} + \hat{\phi}_{0,xy} - \hat{\phi}_1\hat{\gamma}_1, \tag{27}$$

and

$$\hat{\psi}_{1,t} + \hat{\psi}_{1,xy} - \hat{\gamma}_2\hat{\psi}_1 = 0, \qquad \hat{\phi}_{1,t} - \hat{\phi}_{1,xy} + \hat{\gamma}_2\hat{\phi}_1 = 0, \qquad \hat{\gamma}_{2,x} + (\hat{\phi}_1\hat{\psi}_1)_y = 0. \tag{28}$$

From equation (28), it comes that $\{\hat{\psi}_1, \hat{\phi}_1, \hat{\gamma}_2\}$ is a solution of the (2+1)-dimensional CNLERD equation. Thus, the truncated expansion (26) may actually be a BT. A seed solution may be written as follows:

$$\hat{\psi}_1 = \hat{\phi}_1 = 0, \qquad \hat{\gamma}_2 \equiv \hat{\gamma}_2(y, t). \tag{29}$$

This seed solution may stand for a simple case and may be useful for constructing many other solutions. For other seed solutions that may be found, many other classes of solutions may be derived. It is that property of the P-method of constructing various kind of solutions by means of arbitrary functions that makes it potentially and powerfully underlying. These solutions may be given by equation (26) expressed in a truncated form. Due to the arbitrariness of some functions, explicit forms of the solutions may be given, provided to solve analytically or numerically some constraint equations written as NLPD equations. Many examples will be given in the following section while studying the interactions between such structures.

It should be shown that, with the seed solution given by equation (29), the two first equations of the system (13) may be written in another form as follows:

$$\mathcal{A}\mathcal{V}_{0,x} + \mathcal{B}\mathcal{V}_{0,y} + \mathcal{C}\mathcal{V}_0 = 0, \tag{30}$$

where $\mathcal{V}_0 = (\hat{\psi}_0, \hat{\phi}_0)^T$ and

$$\mathcal{A} = \begin{bmatrix} 2g_x g_y & 0 \\ 0 & 2g_x g_y \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} g_x^2 & \frac{\hat{\psi}_0^2}{2} \\ \frac{\hat{\phi}_0^2}{2} & g_x^2 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 2g_x g_t & 0 \\ 0 & -2g_x g_t \end{bmatrix}. \tag{31}$$

Thus, solving equation (30) by means of the characteristics method, it may easily come

$$\mathcal{V}_0 = \mathcal{G}_0\left(x - \int \frac{\mathrm{d}y}{\mathcal{A}^{-1}\mathcal{B}}\right)\exp\left(\int \frac{\mathrm{d}x}{\mathcal{C}^{-1}\mathcal{A}}\right), \tag{32}$$

where $\mathcal{G}_0$ may stand for an arbitrary array function of $\left(x - \int \frac{\mathrm{d}y}{\mathcal{A}^{-1}\mathcal{B}}\right)$ to be determined.

Now, substituting equations (26) and (29) into equation (1) such that

$$\begin{aligned} \hat{\psi}_t &= \frac{D_t\hat{\psi}_0\cdot g}{g^2}, & \hat{\phi}_t &= \frac{D_t\hat{\phi}_0\cdot g}{g^2}, \\ \hat{\psi}_{xy} &= \frac{D_xD_y\hat{\psi}_0\cdot g}{g^2} - \frac{\hat{\psi}_0 D_xD_y g\cdot g}{g^3}, & \hat{\phi}_{xy} &= \frac{D_xD_y\hat{\phi}_0\cdot g}{g^2} - \frac{\hat{\phi}_0 D_xD_y g\cdot g}{g^3}, \\ (\hat{\psi}\hat{\phi})_y &= \frac{D_y(\hat{\psi}_0\hat{\phi}_0)\cdot g}{g^2}, & \hat{\gamma}_x &= \hat{\gamma}_{2,x} + \frac{D_x\hat{\gamma}_1\cdot g}{g^2} + \frac{D_x\hat{\gamma}_0\cdot g}{g^3} + \frac{\hat{\gamma}_0 g_x}{g^3}, \end{aligned} \tag{33}$$

the Hirota's bilinear form corresponding to equation (1) may be expressed as follows:

$$\begin{aligned} &[(D_t + D_xD_y)\hat{\psi}_0\cdot g - \hat{\psi}_0\hat{\gamma}_1]g - (D_xD_y g\cdot g + \hat{\gamma}_0)\hat{\psi}_0 - g^2\hat{\psi}_0\hat{\gamma}_2 = 0, \\ &[(-D_t + D_xD_y)\hat{\phi}_0\cdot g - \hat{\phi}_0\hat{\gamma}_1]g - (D_xD_y g\cdot g + \hat{\gamma}_0)\hat{\phi}_0 - g^2\hat{\phi}_0\hat{\gamma}_2 = 0, \\ &D_x\hat{\gamma}_0\cdot g + D_y(\hat{\psi}_0\hat{\phi}_0)\cdot g + gD_x\hat{\gamma}_1\cdot g = \hat{\gamma}_0 g_x + \hat{\psi}_0\hat{\phi}_0 g_y. \end{aligned} \tag{34}$$

We note that the symbols D_x, D_y and D_t denote Hirota's operators defined by [56, 57]

$$D_t^m D_x^n D_y^l (G\cdot F) = (\partial_t - \partial_{t'})^m(\partial_x - \partial_{x'})^n(\partial_y - \partial_{y'})^l G(x,y,t)F(x',y',t')|_{x=x',y=y',t=t'}. \tag{35}$$

Equation (34) may be decoupled into the following bilinear equations:

$$\begin{aligned} &(D_t + D_xD_y - \nu)\hat{\psi}_0\cdot g - \hat{\psi}_0\hat{\gamma}_1 = 0, \\ &(-D_t + D_xD_y - \nu)\hat{\phi}_0\cdot g - \hat{\phi}_0\hat{\gamma}_1 = 0, \\ &\nu - \hat{\gamma}_2 - \mu = 0, \qquad \hat{\gamma}_0 g_x + \hat{\psi}_0\hat{\phi}_0 g_y = 0, \\ &(D_xD_y - \mu)g\cdot g + \hat{\gamma}_0 = 0, \qquad D_x\hat{\gamma}_1\cdot g + \delta\hat{\gamma}_0 + \varrho\hat{\psi}_0\hat{\phi}_0 = 0 \\ &D_x\hat{\gamma}_0\cdot g + D_y\left(\hat{\psi}_0\hat{\phi}_0\right)\cdot g = \delta\hat{\gamma}_0 g + \varrho\hat{\psi}_0\hat{\phi}_0 g, \end{aligned} \tag{36}$$

where μ, ν, δ and ϱ may stand for arbitrary constants to be determined. In particular, from equations (26) and (36), it may come

$$\hat{\psi}\hat{\phi} = \frac{g_x(D_xD_y - \mu)g\cdot g}{g_y g^2}. \tag{37}$$

Now, expanding the functions g, $\hat{\psi}_0$, $\hat{\phi}_0$, $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ as power series, and using them in equation (36), we may construct the N-soliton solutions in the usual way. However, we shall not follow this route throughout this paper.

We have checked that the above Hirota's bilinearization holds good for the previous truncation. We may now conclude that the (2+1)-dimensional CNLERD equation is completely integrable. It seems worth investigating the different patterns that may be solutions of this coupled equation. These solutions may be closely related to the arbitrariness of g and $\hat{\gamma}_2$ as we can see from equation (27). In other words, with some suitable choices of g and $\hat{\gamma}_2$, we may solve the NLPD equation (27) in other to find $\hat{\psi}_0$ and $\hat{\phi}_0$.

Throughout the following sections, we shall pay attention to the quantities $|\hat{\psi}\hat{\phi}|$ and $\hat{\gamma}$, which may be expressed as follows:

$$|\hat{\psi}\hat{\phi}| = 2(\partial_x \ln|g|)^2, \qquad \hat{\gamma} = \hat{\gamma}_2 - \frac{D_x D_y g \cdot g}{g^2}, \tag{38}$$

derived from equations (4), (26) and (29). It seems worth noting that this expression of $\hat{\gamma}$ may be related to the universal formula of the MLVSA method [38]. In fact, from a seed solution to equation (28), as given by equation (29), it may easily be seen that this solution may not depend on whether $\hat{\gamma}_1$ is different to zero or not. Thus, if $\hat{\gamma}_1 = 0$, then, from the third equation of the system (13), it may come $g_{x,y} = 0$ which shows that g may be the sum of two arbitrary functions $g_1 \equiv g_1(x,t)$ and $g_2 \equiv g_2(y,t)$. Now, considering instead the case where $\hat{\gamma}_1 \neq 0$, and searching for a class of solutions generalizing the previous ones such that $\hat{\gamma}_1 = \hat{f}_1(x,t)\hat{f}_2(y,t)$ with $\hat{f}_1(x,t)$ and $\hat{f}_2(y,t)$ being arbitrary functions, from the third equation of the system (13), the function g may be expressed as follows:

$$g = a_0 + a_1 p + a_2 q + a_3 pq, \tag{39}$$

where $p = p(x,t)$ and $q = q(y,t)$ may stand for arbitrary functions and the parameters a_i $(i = 0, 1, 2, 3)$ being arbitrary constants. With the above form of g given by equation (39), we may combine the two equations (27) and (32) in order to get a nonlinear system in terms of $\mathcal{G}_0$ and $\hat{\gamma}_2$ which may be solved analytically or numerically to determine $\mathcal{G}_0$ with respect to some arbitrary expressions of $\hat{\gamma}_2$. For simplicity, it may be interesting to take $\hat{\gamma}_2 = 0$ as considered in the following section while investigating the scattering behavior of some localized excitations.

4. Scattering behavior: elastic and nonelastic interaction

First of all, it seems important to present the asymptotic behavior of the localized excitations produced from equations (38) and (39).

Recently, Tang *et al* [49] have studied the elastic and nonelastic interaction between saddle-type ring solitons and peakons. Also, Lou [41] has proposed a method to construct compact solitary waves and compactons on the basis of the universal formula from the MLVSA method, and this method has been extended to generate (2+1)-dimensional solitary waves and solitons namely plateau-type, basin-type and bowl-type ring solitons for the (2+1)-dimensional sine-Gordon equation [58]. Besides, Tang and Lou [59] have provided an interesting way for construction of foldons (folded solitary waves with solitonic properties).

We may assume that $F_i(\omega_i) \equiv F_i(\xi - v_i t) \equiv \int P_i \mathrm{d}x|_{\omega_i \to \pm\infty} \to F_i^{\pm}$ and ω_i invariant as $t \to \infty$. We consider that q and a_i $(i = 0, \ldots, 3)$ are time independent. At the ith excitation, the interaction properties among the localized excitations may be described by the following equations:

$$\begin{aligned}
&\hat{\gamma}_{t\to\mp\infty} \to \sum_{i=1}^{N} 2\frac{A_i}{(a_0 + a_2 q(y) + [a_1 + a_3 q(y)][F_i(\omega_i) + \Omega_i^{\mp}])^2} \\
&|\hat{\psi}\hat{\phi}|_{t\to\mp\infty} \to \sum_{i=1}^{N} 2\left[\frac{P_i(\omega_i)(a_1 + a_3 q(y))}{a_0 + a_2 q(y) + [a_1 + a_3 q(y)]\left[F_i(\omega_i) + \Omega_i^{\mp}\right]}\right]^2 \\
&x|_{t\to\mp\infty} \to \xi + \Gamma_i^{\mp} + X_i(\xi - v_i t),
\end{aligned} \tag{40}$$

where

$$\begin{aligned} A_i &= \big[(a_1+a_3q)\big(a_2+a_3\big[F_i(\omega_i)+\Omega_i^{\mp}\big]\big)\\ &\quad\times\big(a_0+a_2q+[a_1+a_3q(y)]\big[F_i(\omega_i)+\Omega_i^{\mp}\big]\big)-a_3\big]q_yP_i(\omega_i)\\ \Omega_i^{\mp} &= \sum_{j<i}F_j^{\mp}+\sum_{j>i}F_j^{\mp}, \qquad \Gamma_i^{\mp}=\sum_{j<i}G_j^{\mp}+\sum_{j>i}G_j^{\mp}. \end{aligned} \tag{41}$$

Thus, the ith excitation preserves its shape if $\Omega_i^+ = \Omega_i^-$ and its total phase shift is $\Gamma_i^+ - \Gamma_i^-$. Therefore, in order to construct completely elastic interaction properties, it is suggested to select suitable localized functions F_i (and then P_i) such that $\Omega_i^+ = \Omega_i^- (i = 1, \dots, N)$. Multiple (2+1)-dimensional localized solitonic excitations with completely elastic interaction properties may then be built-up from the (1+1)-dimensional multiple localized excitations, provided the above properties are observed. For example, we may derive multiple folded solitary waves ($\Omega_i^+ \neq \Omega_i^-$, at least for one i) or multiple foldons ($\Omega_i^+ = \Omega_i^-$ for all i) from the (1+1)-dimensional localized multivalued functions generating loop solitons [60–68].

As illustrations, we may consider interactions among typical peakons and bubbles, saddle-type ring solitons and compactons. As a result, it is found that depending on some suitable choices of p and q, these interactions may be elastic or inelastic. Thus, in figures 1 and 2, we consider this general setting

$$\begin{aligned} p_x &= a\,\mathrm{sech}^2(\xi)+b\,\mathrm{sech}^2(\xi-vt), \qquad & x &= \xi+\alpha\tanh(\xi)+\beta\tanh(\xi-vt)\\ q_y &= \mathrm{sech}^2(\eta), & y &= \eta+c\tanh(\eta), \end{aligned} \tag{42}$$

where $a, b, c,$ α and β are arbitrary constants. According to their different values, we may construct elastic interaction between bubbles (or foldons), peakons, bell-like solitons, just to name a few. In figure 1, the two initial bubbles whose crests are localized at ($x = -5.725$, $y = 0$, $\hat{\gamma} = 0.001\,14$) and ($x = 0.5$, $y = 0$, $\hat{\gamma} = 0.001\,679$) interact elastically in such a way that they overlap three times before being shifted. The same observation is made for the peakons and bell-like semi-foldons depicted in figure 2.

More investigations may also be made on other type of solitons. Thus, in figure 3 we have studied the scattering between two single saddle rings given by $\hat{\gamma}$, and also scattering between another typical duck-beak solitons given by $|\hat{\psi}\hat{\phi}|$. The detail is given in the caption of figure 3. We may observe that these structures retain their shape after interaction in such a way that the interactions between these structures are clearly elastic. We also consider another typical peakons. The detail on this localized excitation is provided in the caption of figure 4. As a result, the initial peakons interact elastically such that after the interaction, they retain their initial shapes. However, there is some strange phenomenon which occurs when, before the scattering, the previous peakons initially exchange their initial position. Indeed, the small peakon located at ($x = -5$, $y = 0$) interacts attractively with the large one located at ($x = 15$, $y = 0$) such that before the scattering, the two structures maintain their initial properties (shapes, velocities, etc). During the head-on, they overlap together to a single peakon and after the interaction, the amplitude of the small peakon is considerably mitigated on contrary to the large peakon. This situation may lead to a further disappearance of the small peakon. This phenomenon may be worth investigating in many systems since applications may be found in many fields such as biology with gene transmissions, chemistry with ion motions and dynamic changes of color of solutions, physics with nonlinear optics and soliton theory.

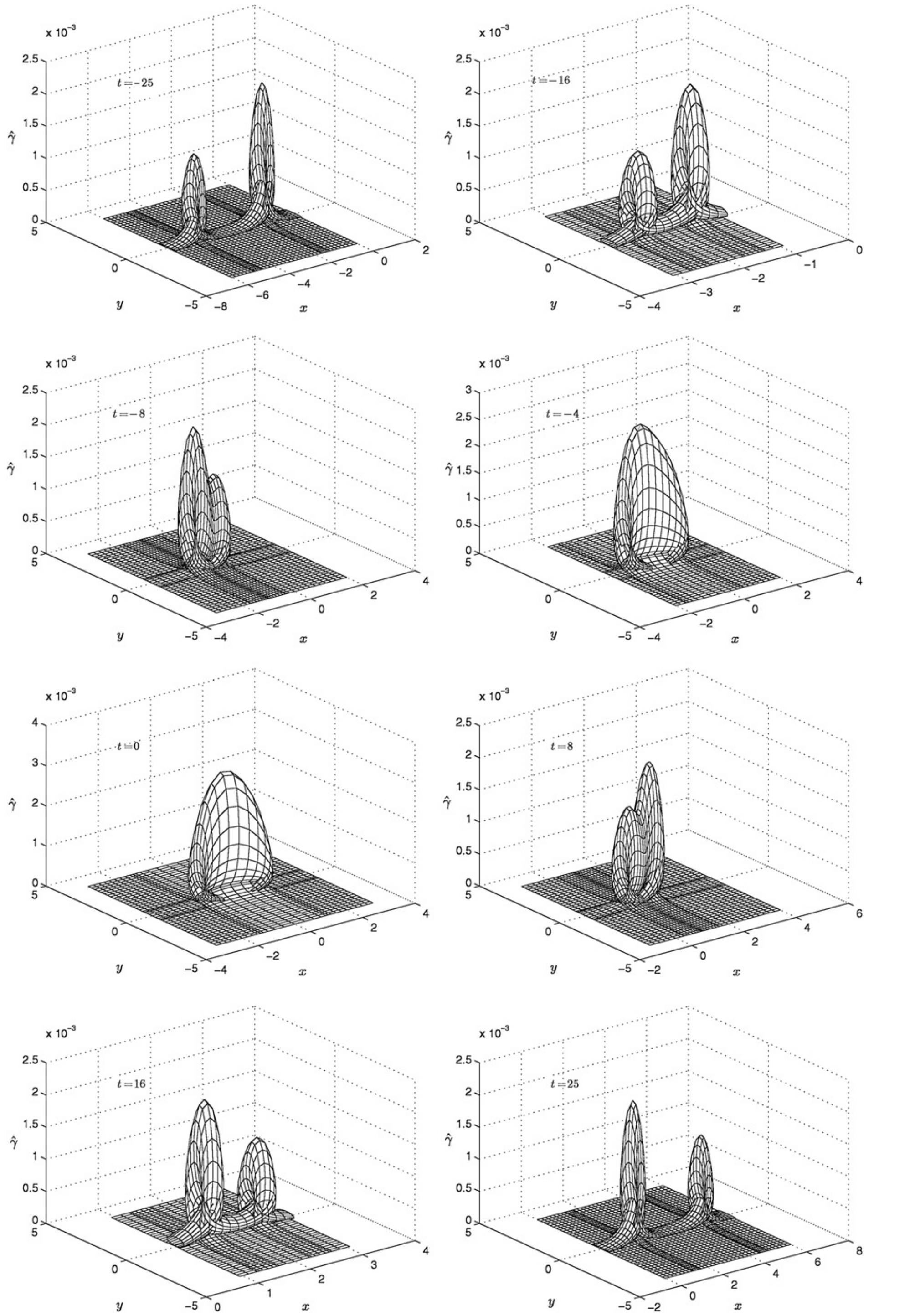

Figure 1. Typical bubble depicted by $\hat{\gamma}$ under the following selection: $a = 0.8$, $b = 0.5$, $c = -2$, $\alpha = -1.5$, $\beta = -1.5$, $v = 0.25$, $a_0 = 20$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1/35$ and $\hat{\gamma}_2 = 0$. The plots are depicted from upper panel to lower panel of the figure at times $t = -25$, $t = -16$, $t = -8$, $t = -4$, $t = 0$, $t = 8$, $t = 16$ and $t = 25$.

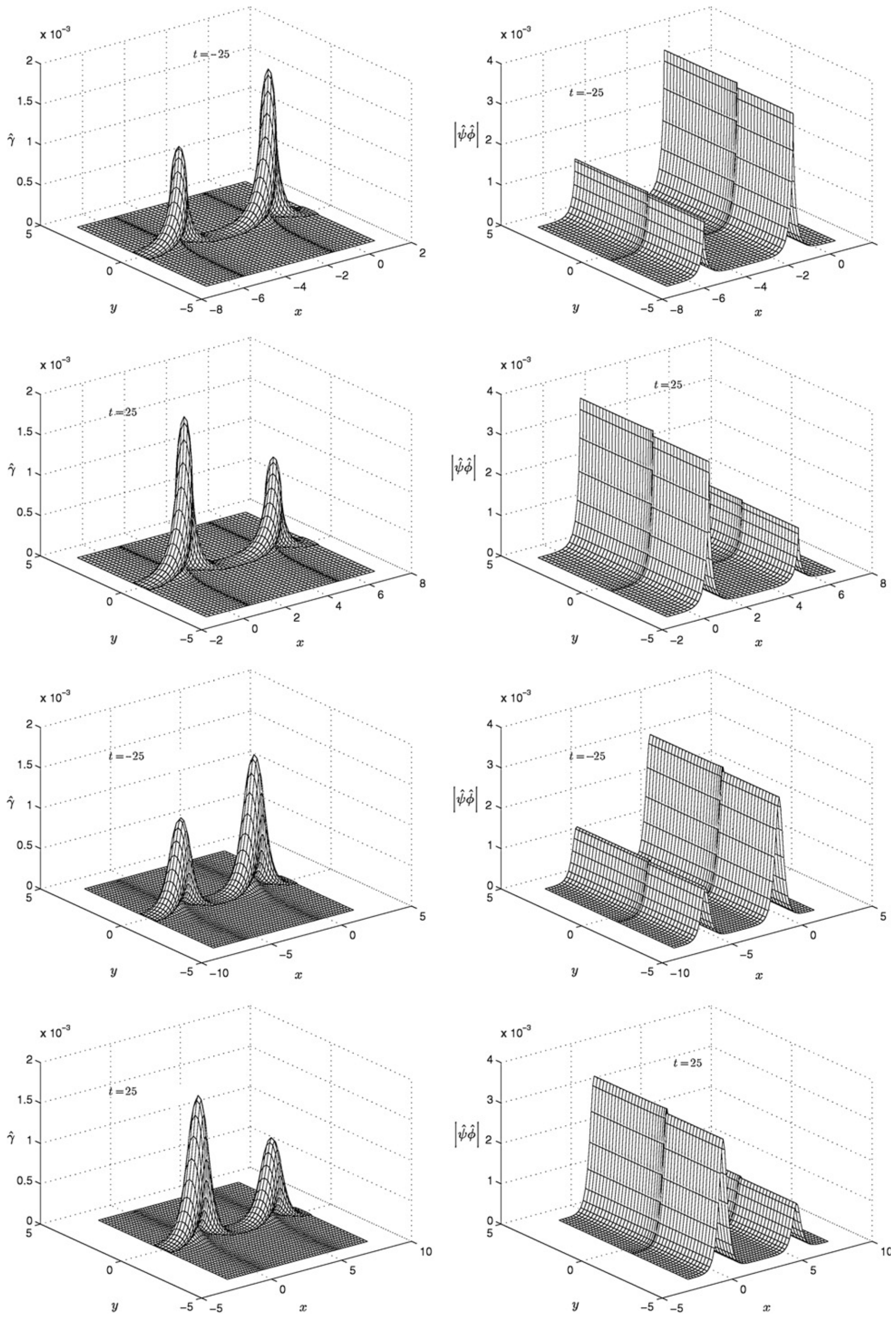

Figure 2. Elastic interaction among typical peakon and bell-like semifoldons depicted by $\hat{\gamma}$ and $|\hat{\psi}\hat{\phi}|$ at times $t = -25$ and $t = 25$ under the following selection: in the case of peakon semi-foldon (the upper four panels), $a = 0.8$, $b = 0.5$, $c = -2$, $\alpha = -1$, $\beta = -1$, $v = 0.25$, $a_0 = 20$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1/35$ and $\hat{\gamma}_2 = 0$. In the case of bell-like semifoldon (the lower four panels), $a = 0.8$, $b = 0.5$, $c = -2$, $\alpha = -1.5$, $\beta = -1.5$, $v = 0.25$, $a_0 = 20$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1/35$ and $\hat{\gamma}_2 = 0$.

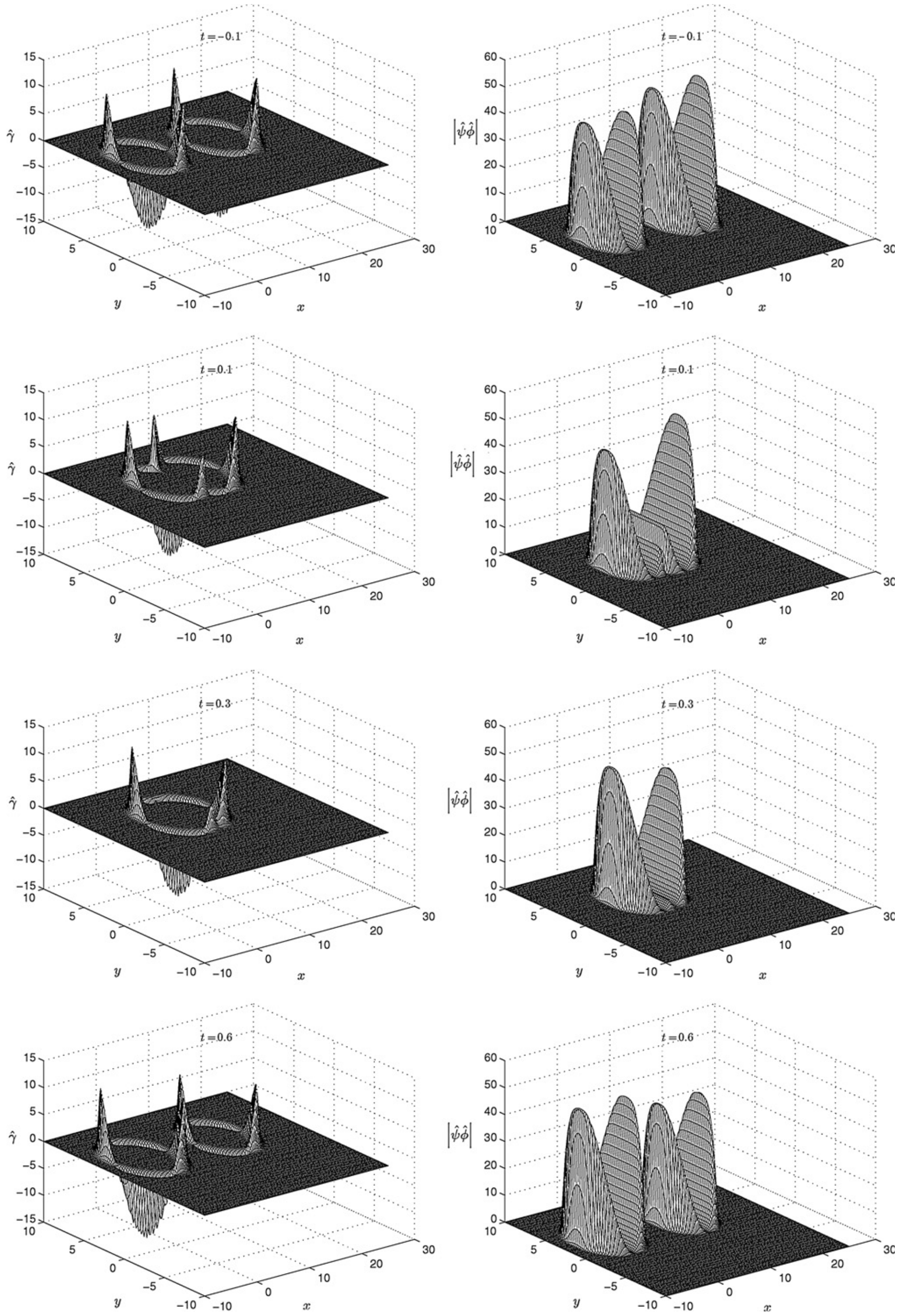

Figure 3. Elastic interaction among typical saddle ring and duck-beak-type solutions depicted by $\hat{\gamma}$ and $|\hat{\psi}\hat{\phi}|$, respectively, at times $t = -0.1, t = 0.1, t = 0.3$ and $t = 0.6$ under the following selection: $p = \exp[-(4(x-20t)/5)^2+15]/5+\exp[-3/5(x+20t-19/2)^2+17]/5$, $q = \exp(-y^2)/5 + \exp[(y-8/3)^2]$. $a_0 = a_1 = a_2 = 1, a_3 = 0$ and $\hat{\gamma}_2 = 0$.

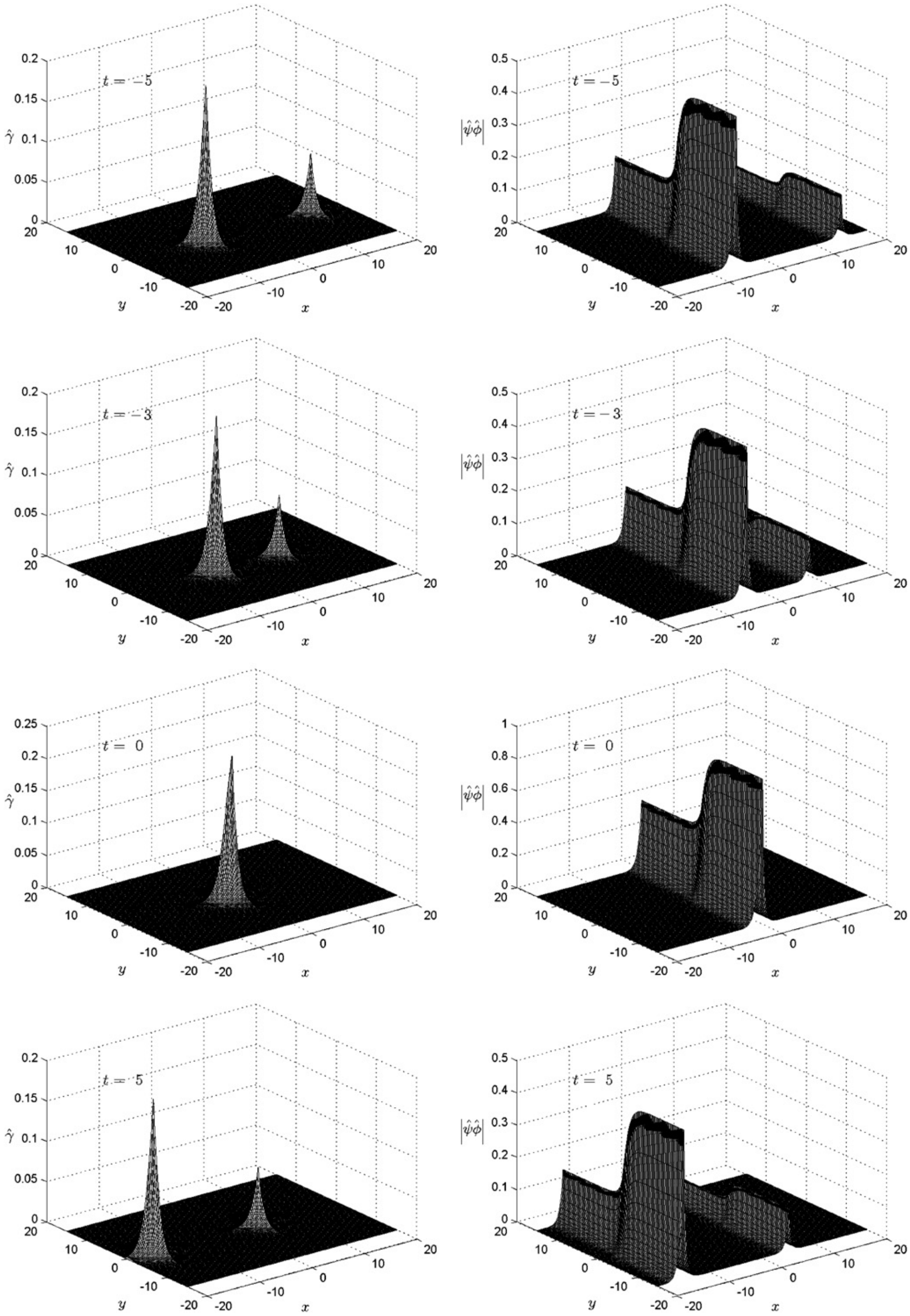

Figure 4. Elastic interaction among typical peakons depicted by $\hat{\gamma}$ and $|\hat{\psi}\hat{\phi}|$ under the following selection: $p_{1x} = 1/\sinh(-|1 - x + t|)$, $p_{2x} = 1/\sinh(-|1 - x + t|)$ such that $p_x = p_{1x} + p_{2x}$ and $q_y = 1/\sinh(-|1 - y|)$. $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1/9$ and $\hat{\gamma}_2 = 0$. The plots are depicted from upper stage to lower stage of the figure at times $t = -5$, $t = -3$, $t = 0$ and $t = 5$.

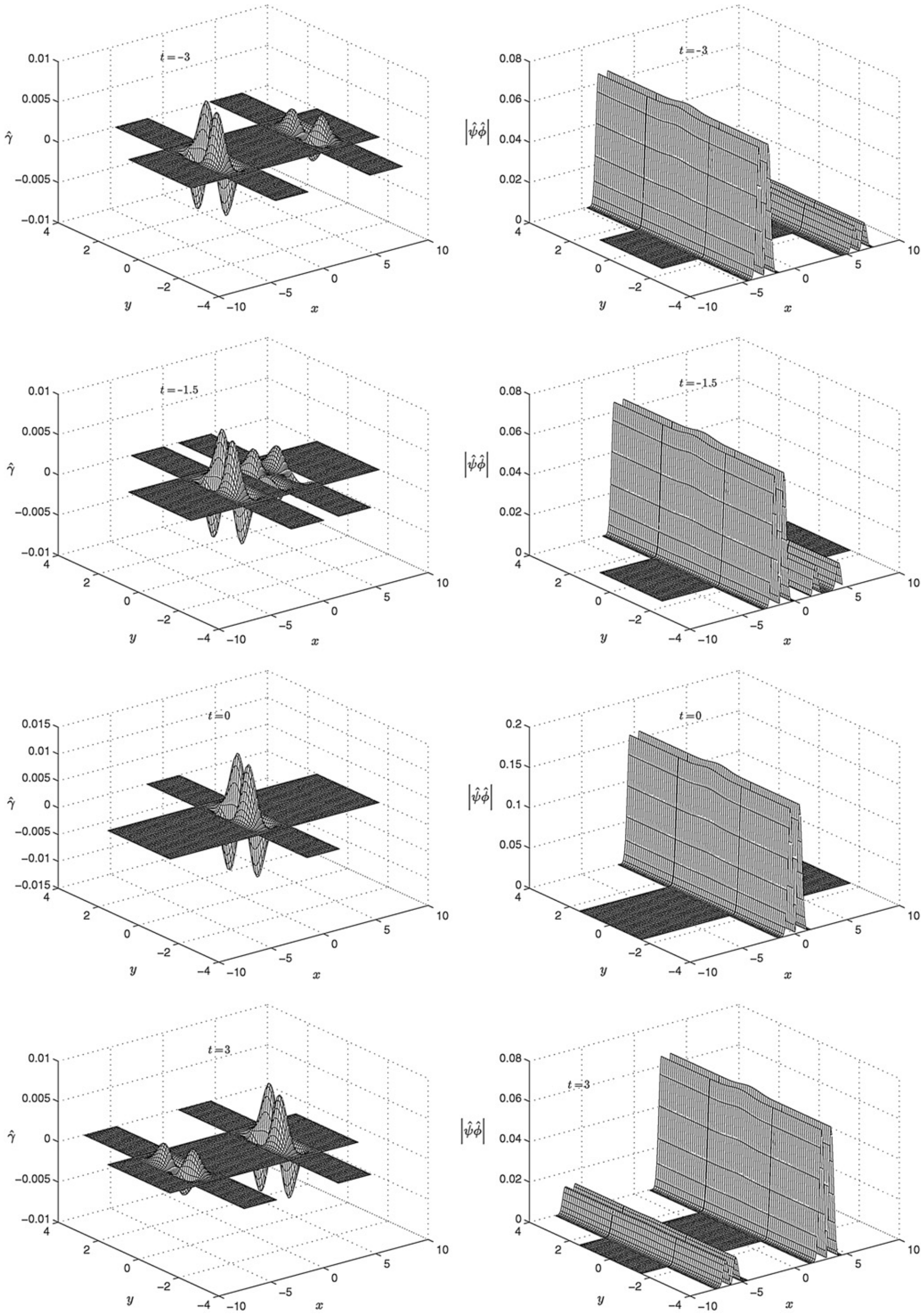

Figure 5. Elastic interaction among typical compactons depicted by $\hat{\gamma}$ and $|\hat{\psi}\hat{\phi}|$ under the following selection: $R_1^p(\varphi_1) = -2\cos^5(\varphi_1)$, $R_2^p(\varphi_2) = -\cos^5(\varphi_2)$ and $R^q(\phi) = \cos^5(\phi)$ with $\varphi_1 = x - t, \varphi_2 = x + 2t, \phi = y, \varphi_{11} = \varphi_{12} = \phi_1 = -\pi/2$, and $\varphi_{21} = \varphi_{22} = \phi_2 = \pi/2$. $a_0 = 17, a_1 = 1, a_2 = 1, a_3 = 1/14$ and $\hat{\gamma}_2 = 0$. The plots are depicted from upper stage to lower stage of the figure at times $t = -3, t = -1.5, t = 0$ and $t = 3$.

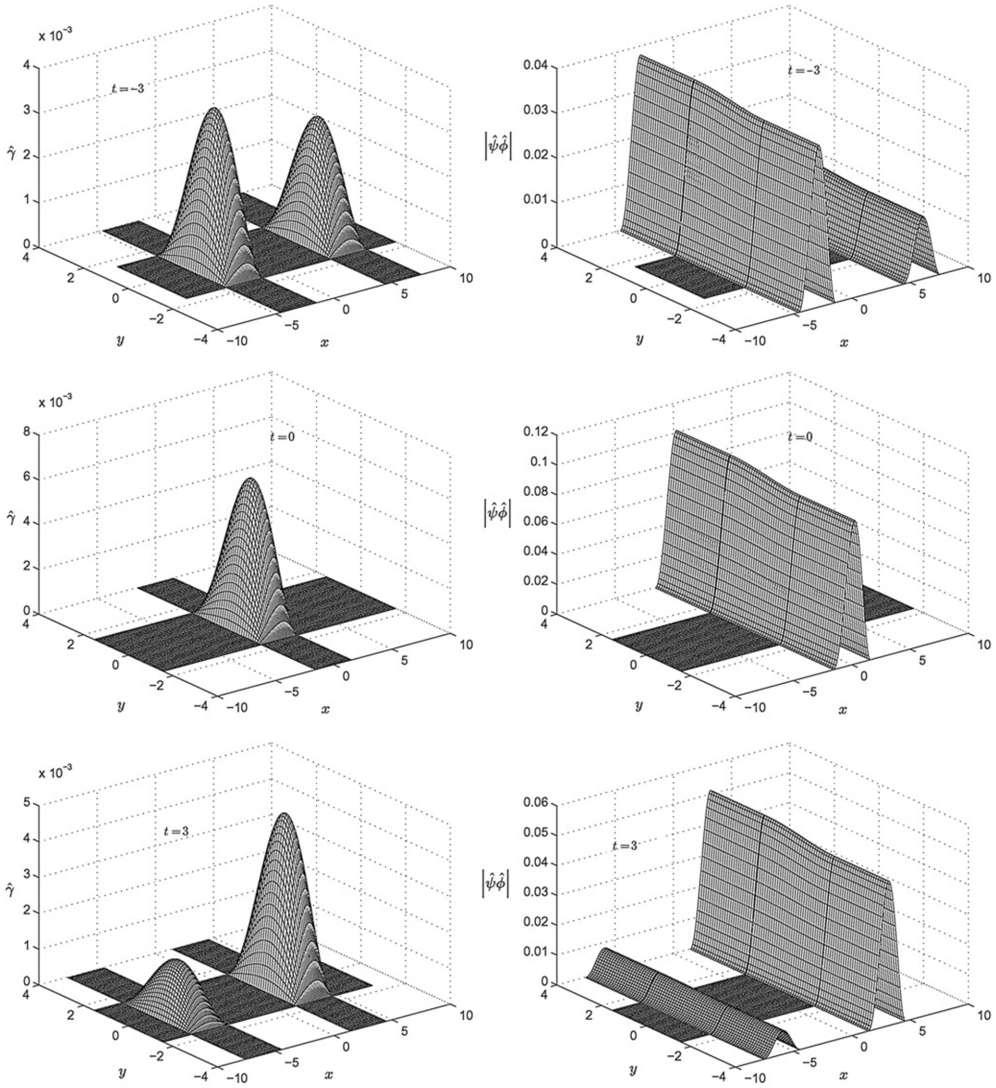

Figure 6. Inelastic interaction among some typical compactons depicted by $\hat{\gamma}$ and $|\hat{\psi}\hat{\phi}|$ under the following selection: $R_1^p(\varphi_1) = -2\sin(\varphi_1) - 2$, $R_2^p(\varphi_2) = -\sin(\varphi_2) - 1$ and $R^q(\phi) = \sin(\phi) + 1$ with $\varphi_1 = x - t$, $\varphi_2 = x + 2t$, $\phi = y$, $\varphi_{11} = \varphi_{12} = \phi_1 = -\pi/2$, and $\varphi_{21} = \varphi_{22} = \phi_2 = \pi/2$. $a_0 = 17$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1/14$ and $\hat{\gamma}_2 = 0$. The plots are depicted from upper stage to lower stage of the figure at times $t = -3$, $t = 0$ and $t = 3$.

Another interesting excitation is the compactons which may be derived from the following equation [69]

$$p = \sum_{i=1}^{M} \begin{cases} 0, & \varphi_i \leqslant \varphi_{1i}, \\ R_i^p(\varphi_i) - R_i^p(\varphi_{1i}), & \varphi_{1i} < \varphi_{2i}, \\ R_i^p(\varphi_{2i}) - R_i^p(\varphi_{1i}), & \varphi_{2i} < \varphi_i, \end{cases} \qquad q = \sum_{i=1}^{N} \begin{cases} 0, & \phi_i \leqslant \phi_{1i}, \\ R_i^q(\phi_i) - R_i^q(\phi_{1i}), & \phi_{1i} < \phi_{2i}, \\ R_i^q(\phi_{2i}) - R_i^q(\phi_{1i}), & \phi_{2i} < \phi_i, \end{cases} \tag{43}$$

where $\varphi_i = x - c_i t (i = 1, \ldots, M)$ and $\phi_i = y - v_i t (i = 1, \ldots, M)$, c_i and v_i being arbitrary constants standing for velocities of waves. $R_i^p(\varphi_i)(i = 1, \ldots, M)$ and $R_i^q(\phi_i)(j = 1, \ldots, N)$ may be differentiable functions yielding many kind of (2+1)-dimensional compactons. In general, interaction among compactons may be elastic or not [41]. We try to see whether this remark still applies to our system under investigation. We then consider two kinds of compactons. The interaction among the first kind is clearly shown in figure 5 where detail is provided in the caption. The interaction is utterly elastic in such a way that the two initial structures retain their shapes after scattering. The second kind of compactons may be given by $R_1^p(\varphi_1) = -2\sin(\varphi_1) - 2$, $R_2^p(\varphi_2) = -\sin(\varphi_2) - 1$ and $R^q(\phi) = \sin(\phi) + 1$ with $\varphi_1 = x - t, \varphi_2 = x + 2t, \phi = y, \varphi_{11} = \varphi_{12} = \phi_1 = -\pi/2$, and $\varphi_{21} = \varphi_{22} = \phi_2 = \pi/2$. $a_0 = 17, a_1 = 1, a_2 = 1, a_3 = 1/14$ and $\hat{\gamma}_2 = 0$. As it may be observed in figure 6, the two initial structures interact inelastically in such a way that there is an exchange of amplitudes leading to a decrease of the amplitude of the small structure.

5. Summary

In this paper, we have investigated the P-property of the (2+1)-dimensional CNLERD equation and proved that it is completely integrable. For the existence of the abundant structures of (2+1)-dimensional CNLERD equation, it is quite important but difficult to investigate the interaction properties for all the possible localized excitations. We have presented the interactions of some special types of localized traveling excitations such as saddle ring solitons, duck-beak solitons, peakons, semifoldons, foldons and compactons. For the ring and duck-beak solitons, during the scattering, the two structures pass through each other and completely preserve their shapes, velocities and phases. Thus, they interact elastically. The same phenomenon is witnessed in the case of semifoldons, foldons such as bubbles. For the traveling peakons, there may exist some types which during the elastic interaction process, the peakons completely exchange their shapes (see figure 4). This kind of peakons is derived from piecewise functions. However, when the initial positions of the small and large peakons are interchanged, the interaction becomes nonelastic owing to the considerable decrease of the amplitude of the small peakon after interactions. For the traveling compactons, the interactions may be elastic as previously observed in the case of ring solitons. Since we may construct a rich variety of compactons from piecewise functions, some types may be found which may not interact elastically. For the foldons (say bubbles), the interaction process has a particular feature. Indeed, during the interactions, the initial bubbles attract each other and overlap three times before being shifted with preserved shapes. It is important to mention here that if a NLPD equation possesses soliton-like solutions, then there may not be any doubt concerning its integrability. Nonetheless, all integrable NLPD equations may not possess solitonic structures even though, due to their integrability properties, they may possess an infinite number of conserved quantities [41, 69–72].

Moreover, there is still too much to deal with interactions in (2+1)-dimensional systems. Thanks to the arbitrariness of $p(x, t)$ and $q(y, t)$. We consider the following system expressing p and q such that

$$p = \alpha \tanh(k_1 x + \omega_1 t) + \beta \tanh^3(k_2 x + \omega_2 t)/2, \qquad q = \gamma \tanh(k_3 y), \qquad (44)$$

which in fact, has been recently investigated by Radha and Lou [73] in the (2+1)-dimensional generalized Sasa–Satsuma equation. From these studies, it has been pointed out inelastic interaction among multiple dromion solutions. In fact, when the amplitude of one of the dromions is relatively small before the scattering takes place, the inelastic interactions may be considered as approximate dromion fission and the amplitude of the split dromions is the same as that of the original one. This phenomenon has also been witnessed in the (2+1)-dimensional CNLERD equation through two typical solitons, the previous dromions described by $\hat{\gamma}$ and a sech-like soliton described by $|\hat{\psi}\hat{\phi}|$, with the following parameters: $\alpha = 18, \beta = 1/2, \gamma = 1/3, k_1 = 1/2, k_2 = 1, k_3 = 1/3, \omega_1 = -2$ and $\omega_2 = 2$ with $a_0 = 20, a_1 = 1, a_2 = 1, a_3 = 1/50$ and $\hat{\gamma}_2 = 0$. Further, when the amplitude of the dromions is relatively small after interaction, then the inelastic interaction is considered as approximate dromion fusion such that the velocity of the fused dromion is the same as that of one of the original dromions. The same phenomenon also occurs in the (2+1)-dimensional CNLERD equation while trying instead the following parameters: $\alpha = 1/2, \beta = 18, \gamma = 1, k_1 = 1, k_2 = 2, k_3 = 1/2, \omega_1 = 2$ and $\omega_2 = -2$ with $a_0 = 20, a_1 = 1, a_2 = 1, a_3 = 1/50$ and $\hat{\gamma}_2 = 0$. For some convenience, these plots have not been depicted in this paper, but are worth noting in the area of survey of different types of scattering phenomena among such topological structures. It may be worthy to mention that not only localized excitations found in this paper are solutions of the (2+1)-dimensional CNLERD, but periodic waves may also be found. Thus, further interests may be paid on such structures. Although the standard WTC's P-expansion method used in this paper may stand for a powerful method among other techniques in investigating integrability properties and in finding some exact solutions to nonlinear systems, it may not be possible to find some physically significant nonsingular localized solutions to some model systems. Thus, Conte [74] has developed an alternative P-analysis approach, the invariant P-analysis for such models. A modification of the truncated Conte's expansion has been presented by Pickering [75, 76], and generalized further by Lou [77–79]. He has shown that the standard and nonstandard truncations of the generalized P-expansion may lead to the construction of new explicit exact solutions. It seems interesting to investigate this generalized method to the (2+1)-dimensional CNLERD equation in a further interest and depict more physically significant solutions.

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